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Matrices coupled in a chain: II. Spacing functions

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Abstract. For the eigenvalues of p complex Hermitian $n \times n$ matrices coupled in a chain, we give a method of calculating the spacing functions. This is a generalization of the one-matrix case which has been known for a long time.

1. Introduction

Let us recall here a few facts concerning the case of one matrix. For an $n \times n$ complex Hermitian matrix A with matrix elements probability density $\exp[-\operatorname{tr} V(A)]$, the probability density of its eigenvalues $\boldsymbol{x} := \{x_1, x_2, \ldots, x_n\}$ is [1]

$$F(\boldsymbol{x}) \propto \exp\left[-\sum_{j=1}^{n} V(x_j)\right] \prod_{1 \leqslant i < j \leqslant n} (x_j - x_i)^2 \tag{1.1a}$$

$$\propto \det[K(x_i, x_j)]_{i,j=1,\ldots,n} \tag{1.1b}$$

where $V(x)$ is a real polynomial of even order, the coefficient of the highest power being positive; $K(x, y)$ is defined by

$$K(x, y) := \exp\left[-\frac{1}{2}V(x) - \frac{1}{2}V(y)\right] \sum_{i=0}^{n-1} \frac{1}{h_i} P_i(x) P_i(y) \tag{1.2}$$

$P_i(x)$ is a real polynomial of degree i and the polynomials are chosen orthogonal with the weight $\exp[-V(x)]$,

$$\int P_i(x) P_j(x) \exp[-V(x)]\, \mathrm{d}x = h_i \delta_{ij}. \tag{1.3}$$

Here and in what follows, all the integrals are taken from $-\infty$ to $+\infty$, unless explicitly stated otherwise.

The correlation function $R_k(x_1, \ldots, x_k)$, i.e. the density of ordered sets of k eigenvalues within small intervals around $x_1, \ldots, x_k$, ignoring the other eigenvalues, is

$$R_k(x_1, \ldots, x_k) := \frac{n!}{(n-k)!} \int F(\boldsymbol{x})\, \mathrm{d}x_{k+1} \ldots \mathrm{d}x_n$$
$$= \det[K(x_i, x_j)]_{i,j=1,\ldots,k}. \tag{1.4}$$

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The spacing function $E(k, I)$, i.e. the probability that a chosen domain I contains exactly k eigenvalues $(0 \leqslant k \leqslant n)$, is

$$E(k, I) := \frac{n!}{k!(n-k)!} \int F(\boldsymbol{x}) \left[\prod_{j=1}^{k} \chi(x_j) \right] \left[\prod_{j=k+1}^{n} [1 - \chi(x_j)] \right] \mathrm{d}x_1 \ldots \mathrm{d}x_n$$
$$= \frac{1}{k!} \left(\frac{\mathrm{d}}{\mathrm{d}z} \right)^k R(z, I) \bigg|_{z=-1} \tag{1.5}$$

where $\chi(x)$ is the characteristic function of the domain I,

$$\chi(x) := \begin{cases} 1 & \text{if } x \in I \\ 0 & \text{otherwise} \end{cases} \tag{1.6}$$

and $R(z, I)$ is the generating function of the integrals over I of the correlation functions $R_k(x_1, \ldots, x_k)$,

$$R(z, I) := \int F(\boldsymbol{x}) \prod_{j=1}^{n} [1 + z\chi(x_j)] \, \mathrm{d}x_j = \sum_{k=0}^{n} \frac{\rho_k}{k!} z^k \tag{1.7}$$

$$\rho_k = \begin{cases} 1 & k = 0 \\ \displaystyle\int R_k(x_1, \ldots, x_k) \prod_{j=1}^{k} \chi(x_j) \, \mathrm{d}x_j & \text{otherwise.} \end{cases} \tag{1.8}$$

The $R(z, I)$ of equation (1.7) can be expressed as a determinant

$$R(z, I) = \det[G_{ij}]_{i,j=0,\ldots,n-1} \tag{1.9}$$

where, using the orthogonality, equation (1.3), of polynomials $P_i(x)$ and splitting the constant and linear terms in z, G_{ij} reads

$$G_{ij} = \frac{1}{h_i} \int P_i(x) P_j(x) \exp[-V(x)][1 + z\chi(x)] \, \mathrm{d}x = \delta_{ij} + \bar{G}_{ij}. \tag{1.10}$$

Finally, $R(z, I)$ can also be written as the Fredholm determinant

$$R(z, I) = \prod_{k=1}^{n} [1 + \lambda_k(z, I)] \tag{1.11}$$

of the integral equation

$$\int N(x, y) f(y) \, \mathrm{d}y = \lambda f(x) \tag{1.12}$$

where, remarkably, the kernel $N(x, y)$ is simply $zK(x, y)\chi(y)$ with $K(x, y)$ of equation (1.2). The $\lambda_i(z, I)$ are the eigenvalues of the above equation and also of the matrix $[\bar{G}_{ij}]$.

These results can be extended to a chain of p complex Hermitian $n \times n$ matrices. We consider the probability density for their elements

$$\mathcal{F}(A_1, \cdots, A_p) \propto \exp[-\mathrm{tr}\{\tfrac{1}{2} V_1(A_1) + V_2(A_2) + \cdots + V_{p-1}(A_{p-1}) + \tfrac{1}{2} V_p(A_p)\}]$$
$$\times \exp[\mathrm{tr}\{c_1 A_1 A_2 + c_2 A_2 A_3 + \cdots + c_{p-1} A_{p-1} A_p\}]. \tag{1.13}$$

Here $V_j(x)$ are real polynomials of even order with positive coefficients of their highest power and the c_j are real constants such that all the integrals which follow converge. For each j the eigenvalues of the matrix A_j are real and will be denoted by $\boldsymbol{x_j} := \{x_{j1},$

$x_{j2}, \ldots, x_{jn}\}$. The probability density for the eigenvalues of all the p matrices resulting from equation (1.13) is [2–5]

$$F(\boldsymbol{x_1}; \ldots; \boldsymbol{x_p}) = C \exp\left[-\sum_{r=1}^{n} \{\tfrac{1}{2} V_1(x_{1r}) + V_2(x_{2r}) + \cdots + V_{p-1}(x_{p-1r}) + \tfrac{1}{2} V_p(x_{pr})\}\right] \times \left[\prod_{1\leqslant r<s\leqslant n} (x_{1s}-x_{1r})(x_{ps}-x_{pr})\right] \det[\mathrm{e}^{c_1 x_{1r} x_{2s}}] \det[\mathrm{e}^{c_2 x_{2r} x_{3s}}] \ldots \det[\mathrm{e}^{c_{p-1} x_{p-1r} x_{ps}}] \tag{1.14}$$

$$= C\left[\prod_{1\leqslant r<s\leqslant n} (x_{1s} - x_{1r})(x_{ps} - x_{pr})\right]\left[\prod_{k=1}^{p-1} \det[w_k(x_{kr}, x_{k+1s})]_{r,s=1,\ldots,n}\right] \tag{1.15}$$

where

$$w_k(x, y) := \exp[-\tfrac{1}{2} V_k(x) - \tfrac{1}{2} V_{k+1}(y) + c_k x y] \tag{1.16}$$

and C is a normalization constant such that the integral of F over all the np variables x_{ir} is one.

The correlation function

$$R_{k_1,\ldots,k_p}(x_{11}, \ldots, x_{1k_1}; \ldots; x_{p1}, \ldots, x_{pk_p}) := \int F(\boldsymbol{x_1}; \ldots; \boldsymbol{x_p}) \prod_{j=1}^{p}\left[\frac{n!}{(n-k_j)!} \prod_{r_j=k_j+1}^{n} \mathrm{d}x_{jr_j}\right] \tag{1.17}$$

was calculated in a previous paper [6] to be an $m \times m$ determinant ($m = k_1 + \cdots + k_p$)

$$R_{k_1,\ldots,k_p}(x_{11}, \ldots, x_{1k_1}; \ldots; x_{p1}, \ldots, x_{pk_p}) = \det[K_{ij}(x_{ir}, x_{js})]_{i,j=1,\ldots,p;r=1,\ldots,k_i;s=1,\ldots,k_j}. \tag{1.18}$$

This is the density of ordered sets of k_j eigenvalues of A_j within small intervals around $x_{j1}, \ldots, x_{jk_j}$ for $j = 1, 2, \ldots, p$. The expression of K_{ij} is recalled at the end of section 2.

Here we will consider the spacing function $E(k_1, I_1; \ldots; k_p, I_p)$, i.e. the probability that the domain I_j contains exactly k_j eigenvalues of the matrix A_j for $j = 1, \ldots, p$, $0 \leqslant k_j \leqslant n$. The domains I_j may have overlaps. As in the one-matrix case one has evidently

$$E(k_1, I_1; \ldots; k_p, I_p) = \frac{1}{k_1!}\left(\frac{\partial}{\partial z_1}\right)^{k_1} \cdots \frac{1}{k_p!}\left(\frac{\partial}{\partial z_p}\right)^{k_p} R(z_1, I_1; \ldots; z_p, I_p)\bigg|_{z_1=\cdots=z_p=-1} \tag{1.19}$$

with the generating function

$$R(z_1, I_1; \ldots; z_p, I_p) = \int F(\boldsymbol{x_1}; \ldots; \boldsymbol{x_p}) \prod_{j=1}^{p}\prod_{r=1}^{n}[1 + z_j \chi_j(x_{jr})]\mathrm{d}x_{jr} \tag{1.20}$$

$$= \sum_{k_1=0}^{n} \cdots \sum_{k_p=0}^{n} \frac{\rho_{k_1,\ldots,k_p}}{k_1! \ldots k_p!} z_1^{k_1} \ldots z_p^{k_p} \tag{1.21}$$

where $\rho_{0,0,\ldots,0} = 1$ and otherwise

$$\rho_{k_1,\ldots,k_p} = \prod_{j=1}^{p}\left[\int_{I_j} \prod_{r=1}^{k_j} \mathrm{d}x_{jr}\right] R_{k_1,\ldots,k_p}(x_{11}, \ldots, x_{1k_1}; \ldots; x_{p1}, \ldots, x_{pk_p}). \tag{1.22}$$

$\chi_j(x)$ being the characteristic function of the domain I_j, equation (1.6).

The function $R(z_1, I_1; \ldots; z_p, I_p)$ will be expressed as an $n \times n$ determinant. It will also be written as a Fredholm determinant, the kernel of which will now depend on the variables $z_1, \ldots, z_p$ and the domains $I_1, \ldots, I_p$ in a more involved way than in the one-matrix case. In particular, it does not have the remarkable form mentioned after equation (1.12).

2. The generating function $R(z_1, I_1; \dots; z_p, I_p)$

The expression of F, equation (1.15), contains a product of determinants. As the product of differences

$$\Delta(\boldsymbol{x_1}) = \prod_{1\leqslant r<s\leqslant n} (x_{1s} - x_{1r}) \tag{2.1}$$

and $\det[w_1(x_{1r}, x_{2s})]$ are completely antisymmetric and other factors in the integrand of equation (1.20) are completely symmetric in the variables $x_{11}, \dots, x_{1n}$, one can replace $\det[w_1(x_{1r}, x_{2s})]$ under the integral sign in equation (1.20) by a single term, say the diagonal one, and multiply by $n!$. This single term is invariant under a permutation of the variables x_{1r} and simultaneously the same permutation on the variables x_{2r}. Therefore, after integration over the x_{1r}, $r = 1, \dots, n$, the integrand, excluding the factor $\det[w_2(x_{2r}, x_{3s})]$, is completely antisymmetric in the variables $x_{21}, \dots, x_{2n}$ and so one can replace the second determinant $\det[w_2(x_{2r}, x_{3s})]$ by a single term, say the diagonal one, and multiply the result by $n!$. In this way, under the integral sign one can replace successively each of the $p-1$ determinants $\det[w_k(x_{kr}, x_{k+1s})]$ by a single term multiplying the result each time by $n!$

$$R(z_1, I_1; \dots; z_p, I_p) = (n!)^{p-1} C \int \Delta(\boldsymbol{x_1})\Delta(\boldsymbol{x_p}) \left[\prod_{j=1}^{p-1}\prod_{r=1}^{n} w_j(x_{jr}, x_{j+1r})\right] \times \left[\prod_{j=1}^{p}\prod_{r=1}^{n} [1 + z_j \chi_j(x_{jr})] \mathrm{d}x_{jr}\right]. \tag{2.2}$$

Recall that a polynomial is called monic when the coefficient of the highest power is one. Also recall that the product of differences $\Delta(\boldsymbol{x})$ can be written as a determinant

$$\Delta(\boldsymbol{x}) = \det[x_i^{j-1}] = \det[P_{j-1}(x_i)] = \det[Q_{j-1}(x_i)] \tag{2.3}$$

where $P_j(x)$ and $Q_j(x)$ are arbitrary monic polynomials of degree j. As usual, we will choose these polynomials real and bi-orthogonal [6]

$$\int P_j(x)(w_1 * w_2 * \cdots * w_{p-1})(x, y) Q_k(y)\, \mathrm{d}x\, \mathrm{d}y = h_j \delta_{jk} \tag{2.4}$$

with the obvious notation

$$(f * g)(x, y) = \int f(x, \xi) g(\xi, y)\, \mathrm{d}\xi. \tag{2.5}$$

The conditions on the weights w_i which ensure the existence and uniqueness of such polynomials are recalled in appendix A.

The normalization constant C is [6],

$$C = (n!)^{-p} \prod_{i=0}^{n-1} h_i^{-1}. \tag{2.6}$$

Now expand the determinant as a sum over $n!$ permutations $(i) := \begin{pmatrix} 0, \dots, n-1 \\ i_1, \dots, i_n \end{pmatrix}$, $\pi(i)$ being its sign,

$$\det[P_{s-1}(x_{1r})] = \sum_{(i)} \pi(i) P_{i_1}(x_{11}) P_{i_2}(x_{12}) \dots P_{i_n}(x_{1n}). \tag{2.7}$$

Doing the same thing for $\det[Q_{s-1}(x_{pr})]$ and integrating over all the np variables x_{jr}; $j = 1, \ldots, p$; $r = 1, \ldots, n$ in equation (2.2), one obtains

$$R(z_1, I_1; \ldots; z_p, I_p) = \frac{1}{n!} \sum_{(i)} \sum_{(j)} \pi(i)\pi(j) G_{i_1 j_1} G_{i_2 j_2} \ldots G_{i_n j_n} = \det[G_{ij}]_{i,j=0,\ldots,n-1} \tag{2.8}$$

where

$$G_{ij} = \frac{1}{h_i} \int P_i(x_1) \left[\prod_{k=1}^{p-1} w_k(x_k, x_{k+1}) \right] Q_j(x_p) \left[\prod_{k=1}^{p} [1 + z_k \chi_k(x_k)] \mathrm{d}x_k \right]. \tag{2.9}$$

When all the z_k vanish, G_{ij} is equal to δ_{ij} as a consequence of the bi-orthogonality, equation (2.4), of the polynomials $P_i(x)$ and $Q_j(x)$. Let us define $\bar{G}_{ij}$ as follows

$$\bar{G}_{ij} := G_{ij} - \delta_{ij} \tag{2.10}$$

so that

$$\bar{G}_{ij} = \frac{1}{h_i} \int P_i(x_1) \left[\prod_{k=1}^{p-1} w_k(x_k, x_{k+1}) \right] Q_j(x_p) \left[\prod_{k=1}^{p} [1 + z_k \chi_k(x_k)] - 1 \right] \left[\prod_{k=1}^{p} \mathrm{d}x_k \right]. \tag{2.11}$$

Any $n \times n$ determinant is the product of its n eigenvalues and therefore one has

$$R(z_1, I_1; \ldots; z_p, I_p) = \prod_{k=1}^{n} [1 + \lambda_k(z_1, I_1; \ldots; z_p, I_p)] \tag{2.12}$$

where the $\lambda_k(z_1, I_1; \ldots; z_p, I_p)$ are the n roots (not necessarily distinct, either real or pairwise complex conjugates, since $\bar{G}_{ij}$ is real) of the algebraic equation in λ

$$\det[\bar{G}_{ij} - \lambda \delta_{ij}] = 0. \tag{2.13}$$

One can always write a Fredholm integral equation with a separable kernel whose eigenvalues are identical to these (cf [7] for the case of $p = 2$ matrices). Indeed, for any eigenvalue λ the system of linear equations

$$\sum_{j=0}^{n-1} \bar{G}_{ij} \xi_j = \lambda \xi_i \tag{2.14}$$

has at least one solution ξ_i, $i = 0, \ldots, n-1$, not all zero. Multiplying both sides of the above equation by $Q_i(x)$, summing over i and using equation (2.11) gives the Fredholm equation

$$\int N(x, x_p) f(x_p)\, \mathrm{d}x_p = \lambda f(x) \tag{2.15}$$

where

$$f(x) := \sum_{\ell=0}^{n-1} \xi_\ell Q_\ell(x) \tag{2.16}$$

$$N(x, x_p) := \sum_{\ell=0}^{n-1} \frac{1}{h_\ell} Q_\ell(x) \int P_\ell(x_1) \left[\prod_{k=1}^{p-1} w_k(x_k, x_{k+1}) \right] \times \left[\prod_{k=1}^{p} [1 + z_k \chi_k(x_k)] - 1 \right] \left[\prod_{k=1}^{p-1} \mathrm{d}x_k \right]. \tag{2.17}$$

Hence, if λ is an eigenvalue of the matrix $[\bar{G}_{ij}]$, it is also an eigenvalue of the integral equation (2.15). Conversely, since the kernel $N(x, x_p)$ is separable and since the polynomials $Q_i(x)$ for $i = 0, \ldots, n-1$ are linearly independent, if λ and $f(x)$ are, respectively an eigenvalue and an eigenfunction of the integral equation (2.15), then $f(x)$ is necessarily of the form

$$f(x) = \sum_{\ell=0}^{n-1} \xi_\ell Q_\ell(x) \tag{2.18}$$

and the ξ_ℓ, $\ell = 0, \ldots, n-1$, not all zero, satisfy equation (2.14). Therefore λ is a root of equation (2.13).

When one considers the eigenvalues of a single matrix anywhere in the chain, disregarding those of the other matrices, everything works as if one is dealing with the one-matrix case and formulae (1.2), (1.5), (1.7) and (1.11) are valid with minor replacements. Similarly, when one considers properties of the eigenvalues of k ($1 \leqslant k \leqslant p$) matrices situated anywhere in the chain, not necessarily consecutive, everything works as if one is dealing with a chain of only k matrices; the presence of other matrices only modifies the couplings.

To say anything more about the general case is difficult.

When $V_j(x) = a_j x^2$, $j = 1, \ldots, p$, then the polynomials $P_j(x)$ and $Q_j(x)$ are Hermite polynomials $P_j(x) = H_j(\alpha x)$, $Q_j(x) = H_j(\beta x)$, the constants α and β depending on the parameters a_j and the couplings c_j. In this particular case the calculation can be pushed to the end (see appendix B).

For the sake of completeness we repeat here the form of the kernels K_{ij}, equation (1.18), from [6]

$$K_{ij}(x, y) := H_{ij}(x, y) - E_{ij}(x, y) \tag{2.19}$$

where

$$\begin{aligned} H_{ij}(x, y) := \sum_{\ell=0}^{n-1} \frac{1}{h_\ell} \int (w_i * w_{i+1} * \cdots * w_{p-1})(x, x_p) Q_\ell(x_p)\, \mathrm{d}x_p \\ \times \int P_\ell(x_1)(w_1 * w_2 * \cdots * w_{j-1})(x_1, y)\, \mathrm{d}x_1 \qquad \begin{cases} 1 \leqslant i < p \\ 1 < j \leqslant p \end{cases} \end{aligned} \tag{2.20}$$

$$H_{pj}(x, y) := \sum_{\ell=0}^{n-1} \frac{1}{h_\ell} Q_\ell(x) \int P_\ell(x_1)(w_1 * w_2 * \cdots * w_{p-1})(x_1, y)\, \mathrm{d}x_1 \qquad 1 < j \leqslant p \tag{2.21}$$

$$H_{i1}(x, y) := \sum_{\ell=0}^{n-1} \frac{1}{h_\ell} \int (w_i * w_{i+1} * \cdots * w_{p-1})(x, x_p) Q_\ell(x_p)\, \mathrm{d}x_p\, P_\ell(y) \qquad 1 \leqslant i < p \tag{2.22}$$

$$H_{p1}(x, y) := \sum_{\ell=0}^{n-1} \frac{1}{h_\ell} Q_\ell(x) P_\ell(y) \tag{2.23}$$

$$E_{ij}(x, y) := \begin{cases} (w_i * w_{i+1} * \cdots * w_{j-1})(x, y) & \text{if } 1 \leqslant i < j \leqslant p \\ 0 & \text{otherwise.} \end{cases} \tag{2.24}$$

Appendix A. Bi-orthogonal polynomials

Given a non-negative weight function $w(x, y)$ we shall assume that all the integrals

$$m_{i,j} := \langle x^i, y^j \rangle = \int w(x, y) x^i y^j \, dx \, dy \tag{A.1}$$

converge. If

$$G_n := \det[m_{i,j}]_{i,j=0,1,\ldots,n} \tag{A.2}$$

is not zero for all non-negative integers $n = 0, 1, 2, \ldots$, then we write

$$P_n(x) := \frac{1}{G_{n-1}} \det \begin{bmatrix} m_{0,0} & m_{0,1} & \ldots & m_{0,n-1} & 1 \\ m_{1,0} & m_{1,1} & \ldots & m_{1,n-1} & x \\ \ldots & \ldots & \ldots & \ldots & . \\ m_{n,0} & m_{n,1} & \ldots & m_{n,n-1} & x^n \end{bmatrix} \tag{A.3}$$

$$Q_n(x) := \frac{1}{G_{n-1}} \det \begin{bmatrix} m_{0,0} & m_{0,1} & \ldots & m_{0,n} \\ m_{1,0} & m_{1,1} & \ldots & m_{1,n} \\ \ldots & \ldots & \ldots & \ldots \\ m_{n-1,0} & m_{n-1,1} & \ldots & m_{n-1,n} \\ 1 & x & \ldots & x^n \end{bmatrix}. \tag{A.4}$$

These $P_n(x)$ and $Q_n(x)$ are monic polynomials of order n. Also

$$\langle P_n(x), y^j \rangle = \frac{1}{G_{n-1}} \det \begin{bmatrix} m_{0,0} & m_{0,1} & \ldots & m_{0,n-1} & m_{0,j} \\ m_{1,0} & m_{1,1} & \ldots & m_{1,n-1} & m_{1,j} \\ \ldots & \ldots & \ldots & \ldots & . \\ m_{n,0} & m_{n,1} & \ldots & m_{n,n-1} & m_{n,j} \end{bmatrix} = 0 \tag{A.5}$$

if $0 \leqslant j < n$, since the determinant on the right-hand side has two identical columns. Similarly, $\langle x^j, Q_n(y) \rangle = 0$, if $0 \leqslant j < n$. This implies that $P_n(x)$ is orthogonal to all polynomials in y of order smaller than n and $Q_n(y)$ is orthogonal to all polynomials in x of order smaller than n. Also

$$\begin{aligned} h_n &:= \langle P_n(x), Q_n(y) \rangle = \langle P_n(x), y^n \rangle = \langle x^n, Q_n(y) \rangle \\ &= \frac{1}{G_{n-1}} \det[m_{i,j}]_{i,j=0,1,\ldots,n} = G_n / G_{n-1}. \end{aligned} \tag{A.6}$$

The polynomials $P_n(x)$ and $Q_n(y)$ are bi-orthogonal. They are also unique, since if there were two polynomials $P_n(x)$ and $R_n(x)$ both monic and orthogonal to all $Q_j(y)$ with $0 \leqslant j < n$, then their difference, a polynomial of order at most $n - 1$ will be orthogonal to $Q_j(y)$ for $0 \leqslant j \leqslant n$. Expressing this polynomial as a linear combination of the $P_j(x)$ and determining the coefficients by orthogonality with the $Q_j(y)$, one sees that it is identically zero.

Appendix B. Quadratic potentials

For $V_j(x) = a_j x^2$, $j = 1, \ldots, p$, setting

$$W_{a,b,c}(x, y) := \exp(-\tfrac{1}{2} a x^2 - \tfrac{1}{2} b y^2 + c x y) \tag{B.1}$$

one obtains according to equation (2.5) the multiplication law

$$(W_{a,b,c} * W_{a',b',c'})(x, y) = \left(\frac{2\pi}{b + a'} \right)^{1/2} W_{a'',b'',c''}(x, y) \tag{B.2}$$

where

$$a'' = a - \frac{c^2}{b+a'} \qquad b'' = b' - \frac{c'^2}{b+a'} \qquad c'' = \frac{cc'}{b+a'}. \tag{B.3}$$

From equation (1.16), $w_k(x, y) = W_{a_k,a_{k+1},c_k}(x, y)$ and repeated use of the above multiplication law yields

$$W(x, y) := (w_1 * w_2 * \cdots * w_{p-1})(x, y) = d \times W_{a,b,c}(x, y) \tag{B.4}$$

where a, b, c and d are constants depending on the parameters $a_1, \ldots, a_p$ and $c_1, \ldots, c_{p-1}$.

The orthogonality relation (2.4) of the polynomials $P_j(x)$ and $Q_j(x)$ takes the form

$$\int P_j(x) W(x, y) Q_k(y)\, \mathrm{d}x\, \mathrm{d}y = h_j \delta_{jk} \tag{B.5}$$

namely the same relation as in the two-matrix case with the weight $W(x, y)$, an exponential of a quadratic form in x and y. It follows that $P_j(x)$ and $Q_j(x)$ are Hermite polynomials of x times a constant

$$P_j(x) = H_j(\alpha x) \qquad \alpha := \left(\frac{ab - c^2}{2b}\right)^{1/2} \tag{B.6}$$

$$Q_j(x) = H_j(\beta x) \qquad \beta := \left(\frac{ab - c^2}{2a}\right)^{1/2} \tag{B.7}$$

$$h_j = \frac{2\pi}{(ab - c^2)^{1/2}} \left(\frac{c}{\sqrt{ab}}\right)^j 2^j j! d. \tag{B.8}$$

The eigenvalue density of the matrix A_1, for example, ignoring the eigenvalues of other matrices, is from equation (1.18)

$$\begin{aligned} R_1(x) = K_{11}(x, x) &= \sum_{j=0}^{n-1} \frac{1}{h_j} P_j(x) \int W(x, y) Q_j(y)\, \mathrm{d}y \\ &= d \left(\frac{2\pi}{b}\right)^{1/2} \mathrm{e}^{-\alpha^2 x^2} \sum_{j=0}^{n-1} \frac{1}{h_j} \left(\frac{c}{\sqrt{ab}}\right)^j H_j^2(\alpha x) \\ &= \frac{\alpha}{\sqrt{\pi}} \mathrm{e}^{-\alpha^2 x^2} \sum_{j=0}^{n-1} \frac{H_j^2(\alpha x)}{2^j j!} \end{aligned} \tag{B.9}$$

which in the large n limit is a semicircle of radius $\sqrt{2n}/\alpha$. Thus, in this particular case of coupled matrices, one recovers Wigner's 'semicircle law' for the eigenvalues of a single matrix.

The kernel of the integral equation (2.15) is given by equation (2.17) with $P_j(x)$, $Q_j(x)$ and h_j as given above. To go further, one has to explicitly take the domains I_j into account.

References

[1] See for example, Mehta M L 1991 *Random Matrices* (San Diego, CA: Academic) ch 3 (here only the case $V(x) = x^2$ is considered, but the same method applies when $V(x)$ is any real polynomial of even order)
[2] See for example, Mehta M L 1991 *Random Matrices* (San Diego, CA: Academic) appendix A.5
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